

# Vacuum polarization around a straight wire carrying a steady current

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## Abstract

Motivated by the example of the superconducting cosmic string which can be a physical representation of a straight wire carrying a steady current, we derive in this case the explicit expressions of the induced vector potential, current density and magnetic field due to the vacuum polarization at the first order in the fine structure constant.

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# 1 Introduction

Cosmic strings could be produced at a phase transition in the early universe and have survived to the present day [1, 2]. Moreover, Witten [3] has shown that cosmic strings can behave like superconductors and carry very large electric current up to  $10^{21}$  A. So, they can generate a very large electromagnetic field.

When the gravitational field is neglected, a static and straight superconducting cosmic string is described in the Minkowski spacetime, outside the core of the string, by an infinite straight wire carrying a steady current. It generates a stationary magnetic field. Recently, Amsterdamski and O'Connor [4] and also Mankiewicz and Zambowicz [5] have studied the vacuum polarization around this superconducting cosmic string. These authors use the results established in the case of a strong but slowly varying magnetic field. However, Amsterdamski and O'Connor [4] have furthermore fulfilled an alternative approximation at the first order in a power series expansion in the strength of the inducing magnetic field.

The present paper is concerned with this last approach of the determination of the vacuum polarization around the superconducting cosmic string. Indeed, when the pair creation is neglected, the induced current resulting from the vacuum polarization in an external current has been determined in general by Serber [6] at the first order in the fine structure constant  $\alpha$ . For a point charge at rest the induced electrostatic potential has been given by Uehling [7] but probably due to the lack of physical motivation, so far as we know, the induced magnetic field has not been calculated for this straight current. Now the straight superconducting cosmic string is an interesting example therefore we will determine the induced vector potential, current

density and magnetic field in the case of an infinite straight wire carrying a steady current.

The plan is as follows. In section 2, we recall the formula giving the induced vector potential. We apply in section 3 this formula in the case of a straight current. We also give the induced current density. We calculate the induced magnetic field in section 4 and we give the screening of the intensity of the current in the vicinity of the straight wire. We add in section 5 some concluding remarks.

## 2 Vacuum polarization in an external steady current

In inertial coordinates  $(t, x^k)$ , the Maxwell equations for a steady current  $j^i$  which is conserved reduce to

$$\Delta A^i = j^i \quad \text{with} \quad \partial_i A^i = 0 \quad (1)$$

where  $A^i$  is the vector potential. The induced current  $\langle j^i \rangle$  due to the vacuum polarization in an external current  $j^i$  has been determined by Serber [6] at the first order in  $\alpha$ . But we use actually the equivalent integral expression given by Schwinger [8]

$$\langle j^i(x^k) \rangle = -\frac{\alpha}{6\pi^2} \int K(|x^k - x'^k|) \Delta j^i(x'^k) d^3 x' \quad (2)$$

where the function  $K$  has the expression

$$K(r) = \frac{1}{r} \int_1^\infty \exp(-2mr\xi) \left( \frac{1}{\xi^2} + \frac{1}{2\xi^4} \right) (\xi^2 - 1)^{1/2} d\xi \quad (3)$$

$m$  being the mass of the electron ( $\hbar = c = 1$ ). The induced vector potential  $\langle A^i \rangle$  is then determined from the Maxwell equations with current (2). We derive immediately the formula

$$\langle A^i(x^k) \rangle = -\frac{\alpha}{6\pi^2} \int K(|x^k - x'^k|) j^i(x'^k) d^3x' \quad (4)$$

### 3 Case of a straight wire carrying a steady current

We consider that the superconducting cosmic string is parallel to the  $x^3$  axis and carries a steady current of intensity  $I$ . The non-vanishing component of the electric current density along the  $x^3$  axis is

$$j^3(x^k) = I\delta(x^1)\delta(x^2) \quad (5)$$

We now introduce the cylindrical coordinates  $(\rho, \varphi, z)$ . The vector potential  $A^i$  generated by current (5) has only the component  $A^z$  which has the expression

$$A^z(\rho) = \frac{I}{2\pi} \ln \frac{\rho}{\rho_0} \quad (6)$$

where  $\rho_0$  is an arbitrary length. By deriving  $A^z$  with respect to  $\rho$ , we obtain the magnetic field whose only non-zero component is  $B_\varphi$  given by

$$B_\varphi(\rho) = \frac{I}{2\pi\rho} \quad (7)$$

The induced vector potential  $\langle A^i \rangle$  is given by formula (4) with current (5) in which we perform the integration with respect to  $x'^1$  and  $x'^2$  to obtain

$$\langle A^z(x^k) \rangle = -\frac{\alpha I}{6\pi^2} \int_{-\infty}^{+\infty} K(\sqrt{\rho^2 + (z - z')^2}) dz' \quad (8)$$

where  $A^z$  depends only on  $\rho$  as expected. By inserting expression (3) of the function  $K$  in (8), we have the double integral

$$\begin{aligned} \langle A^z(\rho) \rangle &= -\frac{\alpha I}{3\pi^2} \int_1^\infty \int_0^\infty \frac{\exp(-2m\sqrt{\rho^2 + u^2}\xi)}{\sqrt{\rho^2 + u^2}} \\ &\quad \times \left(\frac{1}{\xi^2} + \frac{1}{2\xi^4}\right)(\xi^2 - 1)^{1/2} d\xi du \end{aligned} \quad (9)$$

Now, we turn to express  $\langle A^z \rangle$  in terms of elementary functions. For this purpose, we perform the change of variables  $u = \rho \sinh y$  and  $\xi = \cosh x$  in integral (9). We thus get

$$\begin{aligned} \langle A^z(\rho) \rangle &= -\frac{\alpha I}{3\pi^2} \int_0^\infty \int_0^\infty \exp(-2m\rho \cosh x \cosh y) \\ &\quad \times \left(1 + \frac{1}{2 \cosh^2 x}\right) \tanh^2 x dx dy \end{aligned} \quad (10)$$

Now the modified Bessel function  $K_0$  can be expressed as

$$K_0(z) = \int_0^\infty \exp(-z \cosh y) dy \quad (11)$$

therefore we can reduce expression (10) in the form of a simple integral expression involving  $K_0$

$$\langle A^z(\rho) \rangle = -\frac{\alpha I}{3\pi^2} \int_0^\infty K_0(2m\rho \cosh x) \left[1 - \frac{1}{2 \cosh^2 x} - \frac{1}{2 \cosh^4 x}\right] dx \quad (12)$$

However, except the first integral appearing in (12), we have not found the two other integrals in the tables. In appendix A we give expressions (21), (25) and (27) of the necessary integrals. In consequence, we can get an explicit expression for the induced vector potential  $\langle A^i \rangle$  in terms of modified Bessel functions  $K_n$

$$\langle A^z(\rho) \rangle = -\frac{\alpha I}{3\pi^2} \left\{ \frac{1}{2} [K_0(m\rho)]^2 - \frac{5}{6} m\rho K_0(m\rho) K_1(m\rho) \right.$$

$$\begin{aligned}
& -\frac{2}{3}m^2\rho^2[K_0(m\rho)]^2 + \frac{7}{9}m^2\rho^2[K_1(m\rho)]^2 \\
& -\frac{2}{9}m^4\rho^4[K_0(m\rho)]^2 + \frac{2}{9}m^4\rho^4[K_1(m\rho)]^2 \\
& -\frac{2}{9}m^3\rho^3K_0(m\rho)K_1(m\rho) \quad \} \quad (13)
\end{aligned}$$

We see immediately from (13) that the range of the induced vector potential is  $1/2m$ .

To be complete, we also give the induced current density  $\langle j^i \rangle$ . By using the identity

$$z^2 K_0''(z) + z K_0'(z) - z^2 K_0(z) = 0$$

we can derive  $\langle j^z \rangle$  from Maxwell's equations (1). We find

$$\begin{aligned}
\langle j^z(\rho) \rangle &= -\frac{4\alpha I m^2}{3\pi^2} \int_0^\infty K_0(2m\rho \cosh x) \\
&\quad \times [\cosh^2 x - \frac{1}{2} - \frac{1}{2 \cosh^2 x}] dx \quad (14)
\end{aligned}$$

We have verified that integral expression (14) coincides with the result of Amsterdamski and O'Connor [4] after a change of variable. With integrals (21), (23) and (25) given in appendix A, we get the explicit expression of the induced current density in terms of modified Bessel functions

$$\begin{aligned}
\langle j^z(\rho) \rangle &= -\frac{\alpha I m^2}{3\pi^2} \{ [K_1(m\rho)]^2 - 2m\rho K_0(m\rho)K_1(m\rho) \\
&\quad - 2m^2\rho^2[K_0(m\rho)]^2 + 2m^2\rho^2[K_1(m\rho)]^2 \} \quad (15)
\end{aligned}$$

The current density  $\langle j^z \rangle$  vanishes exponentially for large  $\rho$  and diverges as  $\alpha I/3\pi^2\rho^2$  at  $\rho = 0$  since  $K_1(z) \sim 1/z$ .

## 4 Screening of the current by vacuum polarization

The modification due to the vacuum polarization of the external magnetic field (7) occurs mainly near the straight wire. We can directly calculate the induced magnetic field  $\langle B_\varphi \rangle$  by deriving expression (13) with respect to  $\rho$ . Alternatively, we can also derive its integral expression

$$\begin{aligned} \langle B_\varphi(\rho) \rangle = & \frac{2\alpha Im}{3\pi^2} \int_0^\infty K_1(2m\rho \cosh x) \\ & \times [\cosh x - \frac{1}{2 \cosh x} - \frac{1}{2 \cosh^3 x}] dx \end{aligned} \quad (16)$$

where we have use the fact that  $K'_0 = -K_1$ . In appendix A we give expressions (22), (24) and (26) of the necessary integrals. Thus, we can get the explicit expression of the induced magnetic field  $\langle B^i \rangle$  in terms of modified Bessel functions

$$\begin{aligned} \langle B_\varphi(\rho) \rangle = & \frac{\alpha Im}{3\pi^2} \{ K_0(m\rho)K_1(m\rho) + \frac{1}{2}m\rho[K_0(m\rho)]^2 \\ & - \frac{5}{6}m\rho[K_1(m\rho)]^2 + \frac{2}{3}m^2\rho^2 K_0(m\rho)K_1(m\rho) \\ & + \frac{2}{3}m^3\rho^3[K_0(m\rho)]^2 - \frac{2}{3}m^3\rho^3[K_1(m\rho)]^2 \} \end{aligned} \quad (17)$$

The total magnetic field  $B^i$  generated by a straight current is

$$B_\varphi(\rho) = \frac{I}{2\pi\rho} + \langle B_\varphi(\rho) \rangle \quad (18)$$

when the vacuum polarization is taken into account at the first order in  $\alpha$ .

We are now in a position to work out the screening of the source of the magnetic field. This procedure is most significant that the analyse of the induced current density given by (15). From the asymptotic formulas

$$K_0(z) \sim -\gamma - \ln(z/2) \quad \text{and} \quad K_1(z) \sim \frac{1}{z}$$

where  $\gamma$  is Euler's constant, we obtain easily from (17) the asymptotic form of  $\langle B_\varphi \rangle$  for small values of  $m\rho$

$$\langle B_\varphi(\rho) \rangle \sim \frac{\alpha I}{3\pi^2 \rho} \left( -\gamma - \frac{5}{6} + \ln 2 - \ln m\rho \right) \quad (19)$$

With formula (18), we evaluate easily the screening of the intensity the current in the vicinity of the straight wire

$$I_{eff} \approx I \left[ 1 + \frac{\alpha}{3\pi^2} \left( -\gamma - \frac{5}{6} + \ln 2 - \ln m\rho \right) \right] \quad (20)$$

for small values of  $m\rho$ .

## 5 Conclusion

Our main result is the explicit determination of induced quantities: vector potential (13), current density (15) and magnetic field (17), due to the vacuum polarization in an external straight current. However, these results are valid in domains where the magnetic field is not too strong because we have not considered higher order effects in the fine structure constant. In the vicinity of this straight wire we have found asymptotic expansion (19) of the induced magnetic field. For above mentioned reasons, we cannot consider this formula for distances from string too small. Furthermore, vacuum polarization effects resulting from other charged particles with mass greater than the electron mass can eventually occur.



## A Appendix

We write down a general formula [9] that we will be needed in the determination of our integrals

$$\int_0^\infty K_{2\mu}(2z \cosh x) \cosh 2\nu x dx = \frac{1}{2} K_{\mu+\nu}(z) K_{\mu-\nu}(z) \quad \text{for } \Re(z) > 0$$

From this we have immediately the following results

$$\int_0^\infty K_0(b \cosh x) dx = \frac{1}{2} [K_0(b/2)]^2 \quad (21)$$

$$\int_0^\infty K_1(b \cosh x) \cosh x dx = \frac{1}{2} K_0(b/2) K_1(b/2) \quad (22)$$

and after the use of the relation

$$\cosh^2 x = \frac{1}{2} (1 + \cosh 2x)$$

$$\int_0^\infty K_0(b \cosh x) \cosh^2 x dx = \frac{1}{4} [K_0(b/2)]^2 + \frac{1}{4} [K_1(b/2)]^2 \quad (23)$$

The second integral that we have needed can be performed as follows. After an integration by part, it becomes

$$\int_0^\infty \frac{K_0(b \cosh x)}{b^2 \cosh^2 x} dx = \frac{1}{b} \int_0^\infty dx K_1(b \cosh x) \cosh x - \int_0^\infty dx \frac{K_1(b \cosh x)}{b \cosh x}$$

by using the relation  $K'_0 = -K_1$ . By virtue of the recurrence relation

$$z K_{\nu-1}(z) - z K_{\nu+1}(z) = -2\nu K_\nu(z)$$

we get moreover

$$\int_0^\infty \frac{K_1(b \cosh x)}{\cosh x} dx = -\frac{b}{4} [K_0(b/2)]^2 + \frac{b}{4} [K_1(b/2)]^2 \quad (24)$$

Finally by combining these results, we obtain thereby the expression of the second integral

$$\int_0^\infty \frac{K_0(b \cosh x)}{\cosh^2 x} dx = \frac{b}{2} K_0(b/2) K_1(b/2) + \frac{b^2}{4} [K_0(b/2)]^2 - \frac{b^2}{4} [K_1(b/2)]^2 \quad (25)$$

The method of computing the third integral is the same. So, we will not reproduce the calculations. We will have to know the expression of the following integral

$$\begin{aligned} \int_0^\infty \frac{K_1(b \cosh x)}{\cosh^3 x} dx &= -\frac{b^2}{4} K_0(b/2) K_1(b/2) - \frac{3b^3}{32} [K_0(b/2)]^2 \\ &\quad + \frac{b^3}{12} [K_1(b/2)]^2 + \frac{b^3}{96} [K_2(b/2)]^2 \end{aligned}$$

that we can transform in the form

$$\begin{aligned} \int_0^\infty \frac{K_1(b \cosh x)}{\cosh^3 x} dx &= -\frac{b^2}{6} K_0(b/2) K_1(b/2) - \frac{b^3}{12} [K_0(b/2)]^2 \\ &\quad + \frac{b^3}{12} [K_1(b/2)]^2 + \frac{b}{6} [K_1(b/2)]^2 \end{aligned} \quad (26)$$

With this result and after some manipulations we will obtain the expression of the third integral

$$\begin{aligned} \int_0^\infty \frac{K_0(b \cosh x)}{\cosh^4 x} dx &= \frac{b^4}{36} [K_0(b/2)]^2 - \frac{b^4}{36} [K_1(b/2)]^2 \\ &\quad + \frac{b^2}{12} [K_0(b/2)]^2 - \frac{5b^2}{36} [K_1(b/2)]^2 \\ &\quad + \frac{b^3}{18} K_0(b/2) K_1(b/2) + \frac{b}{3} K_0(b/2) K_1(b/2) \end{aligned} \quad (27)$$

## References

- [1] Kibble T W B 1976 *J. Phys. A: Math. Gen.* **9** 1387
- [2] Vilenkin A 1985 *Phys. Rep.* **121** 263
- [3] Witten E 1985 *Nucl. Phys. B* **249** 557
- [4] Amsterdamski P and O'Connor D 1988 *Nucl. Phys. B* **298** 429
- [5] Mankiewicz L and Zembowicz R 1988 *Phys. Lett. B* **202** 493
- [6] Serber R 1935 *Phys. Rev.* **48** 49
- [7] Uehling E A 1935 *Phys. Rev.* **48** 55
- [8] Schwinger J 1949 *Phys. Rev.* **75** 651
- [9] Prudnikov A P, Brychkov Yu A and Marichev O I 1986 *Integrals and Series vol. 2* (Gordon and Breach Science Publishers) p. 358